Is the Tsallis q-mean value instable?

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The recent argue about the existence of an instability in the definition of the mean value appearing in the Tsallis non extensive Statistical Mechanic is reconsidered. Here, it is simply underlined that the pair of probability distributions employed in constructing the instability statement have a discontinuous limit when the number of states tends to infinity. That is, although for an arbitrary but finite number of states W both probability distributions are normalized to the unit, their limits $W->\infty$ do not satisfy the normalization condition and thus are not allowed *escort* probabilities for the q-mean value. However, similar distributions converging to the former ones when a parameter W_o is tending to infinity are defined here. They both satisfy the normalization to the unity in the limit $W \to \infty$. This simple change allows to show that the stability condition becomes satisfied, for whatever large but fixed value of W_o is chosen.

Nowadays, the investigation about the description of the non equilibrium statistical properties of a large number of physical systems is being considered by employing the methods of non extensive Statistical Physics introduced by C. Tsallis [1]. The successes of this theory had been many. The description has been able to predict the deviations from the Boltzman distributions measured for numerous physical systems and motivated an intense research activity seeking of new statistical approaches for non equilibrium systems [1–6].

Recently, however, there had been some argues in the literature([7–11]) supporting the existence of an instability in a basic quantity defining the theory: the special Tsallis mean value (q-mean value) of an arbitrary physical quantity Q as given by

$$\langle Q\rangle_q = \frac{\sum_{i=1}^{W} p_i^q Q_i}{\sum_{i=1}^{W} p_i^q}, \tag{1}$$

$$\sum_{i=1}^{W} p_i = 1, \tag{2}$$

which leads to the usual one in the $q \to 1$ limit $\langle Q\rangle_1 = \sum_i^W p_i Q_i$. In the above formula, W is the number of states in the system when it is a finite number and the p_i are the so called *escort* probabilities.

Basically, those works claim that there exist particular probability perturbations, which are properly defined for finite systems (W being finite), under which the $q-$mean values of the physical quantities in the infinite system limit, $W \to \infty$, show variations that can not be made arbitrarily small, when the perturbations (also taken in the limit of infinite system) of the probabilities are chosen as sufficiently small [7].

In this note we will reconsider the discussion given in Ref. [7]. We will follow the notation of that work and define the two probability distributions $\{p_i\}$and $\{p_i'\}$ for $i = 1, 2, ...W$, for a system having W states. The perturbation of the probability will be defined by the sequence of differences

$$\Delta p_i = p_i - p_i', \quad i = 1, 2, ...W. \tag{3}$$

For starting, it is helpful to state, that the demonstration given in [7] of the "non small" character of the variation of the q-mean value of a given physical quantity Q (under sufficiently small perturbations of the probabilities in the infinite number of states limit) is mathematically correct. However, in spite of this, it should be stressed that the instability result is claimed to be valid for the infinite system ($W \to \infty$). Therefore, it must be pointed out that a necessary requirement for the instability to exist is that in the infinite number states limit $W \to \infty$, both distributions should retain their essential meaning of being proper *escort* probabilities satisfying $\sum_{i=1}^{\infty} p_i = 1$ and $\sum_{i=1}^{\infty} p_i' = 1$. This becomes necessary because the infinite number of state system is well defined, and the limit $W \to \infty$ of also well defined *escort* probabilities should satisfy the normalization condition, in order to be considered as acceptable *escort* probabilities for the infinite systems. The lack of satisfaction of this condition in the instability argue in Ref. [7] is the essential point claimed in the present letter.

In what follows, it will be noticed that this central requirement is drastically violated by the probability distributions considered in Ref. [7]. Further, it will shown that similarly defined distributions, but slightly modified for satisfying the above posed additional condition, directly imply the satisfaction of the stability condition for the infinite system.

The fact that the infinite W limit destroys the well defined character of the distributions defined in Ref. [7], follows form a simple inspection of the formulae for those quantities in the $W \to \infty$ limit. For the case $0 < q < 1$, the limits are

$$p_i = \delta_{i1}, \quad p'_i = (1 - \frac{\delta}{2})\ p_i, \quad i = 1, 2, ..., \infty, \tag{4}$$

and for the case $q > 1$:

$$p_i = 0, \quad p'_i = \frac{\delta}{2}\delta_{i1}, \quad i = 1, 2, ..., \infty. \tag{5}$$

Its is clear from the above expressions that for any value of the parameter δ, only one of the four distributions satisfies the normalization condition for the *escort* probabilities of the infinite system. Therefore, we can conclude that non of the two sets of probability distributions obtained as the limits of the (well defined) finite systems distributions, can be associated to proper perturbations of the Tsallis mean values for the infinite physical system.

Let us now show that the when the distributions for finite systems employed in Ref. [7] are slightly modified to satisfy the condition of leading to allowed *escort* probabilities, then the stability condition is satisfied for the redefined probability perturbations in the infinite system. This will complete the issues aimed to be discussed in this letter.

For this purpose, let us consider, the variations of the $q-$mean value of a physical quantity. The construction will closely follows the one in Ref. [7]. Consider a large finite value of W and another also large number $W_o < W$. The number of states of the system W will be taken in the infinite limit, but W_o, although having an arbitrarily large value will remain to be fixed when $W \to \infty$. The new pairs of probability distribution to be defined will essentially coincide in their analytic expressions with the ones given in Ref. [7] but in which W will be replaced by W_o. The concrete expressions for the case $0 < q < 1$ are

$$p_i \ = \ \delta_{i1}, \quad p'_i = (1 - \frac{\delta}{2}\frac{W_o}{W_o - 1})p_i + \frac{\delta}{2}\frac{\theta(W_o - i)}{W_o - 1}, \quad i = 1, 2, ..., W, \tag{6}$$

$$\theta(n) \ = \ \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}, n - integer, \tag{7}$$

and for $q > 1$:

$$p_i = \frac{(1 - \delta_{i1})\theta(W_o - i)}{W_o - 1}, \quad p'_i = (1 - \frac{\delta}{2})p_i + \frac{\delta}{2}\delta_{i1}, \quad i = 1, 2, ..., W. \tag{8}$$

It can be noted that these definitions by construction, do not depend on the number of the states of the system W. Therefore, all the four distributions satisfy the normalization conditions

$$\sum_{i=1}^{W} p_i = 1, \qquad \sum_{i=1}^{W} p'_i = 1, \tag{9}$$

for arbitrary values of W_o and W. Moreover, these distributions, due to their independence of W, exactly coincide with their limit for $W \to \infty$. Thus, they remain exactly normalized to the unity for the infinite number of states limit, and thus correspond to valid *escort* probabilities for the $q-$mean value when the system already has an infinite number of states. Exactly in the same way as it is derived in Ref. [[7]], the modulus of the difference between the q- mean values evaluated with the probability sequences $\{p_i\}$ and $\{p'_i\}$,for the range $0 < q < 1$, can be directly calculated to be

$$\mid \langle Q \rangle_q - \langle Q \rangle'_q \mid = |Q_1 - \frac{(1 - \frac{\delta}{2})^q Q_1 + \frac{(\frac{\delta}{2})^q}{(W_o - 1)^q}\sum_{i=2}^{W_o} Q_i}{(1 - \frac{\delta}{2})^q + (\frac{\delta}{2})^q (W_o - 1)^{1-q}}| \ . \tag{10}$$

This expression is W independent as noted before, then its $W \to \infty$ limit coincides with it. Now, it is clear that the above formula implies that $\mid \langle Q \rangle_q - \langle Q \rangle'_q \mid$ can be fixed to be smaller than any arbitrarily small quantity ϵ after also selecting δ to be sufficiently small for any arbitrarily large but fixed constant W_o. Therefore, the slight redefinition of the densities for to be well defined *escort* probabilities for $W \to \infty$ had allowed to satisfy the stability condition.

Finally, in a similar way, the perturbation of the mean values for the $q>1$ case, can be expressed in identical form as in Ref. [7] but by replacing W by W_o , to get the expression

$$| \langle Q\rangle_q - \langle Q\rangle'_q | = |\frac{\overline{Q}-Q_1}{W_o-1} - \frac{\frac{(1-\frac{\delta}{2})^q}{(W_o-1)^q}(W_o\overline{Q}-Q_1)+(\frac{\delta}{2})^q Q_1}{(\frac{\delta}{2})^q+(1-\frac{\delta}{2})^q(W_o-1)^{1-q}}|, \tag{11}$$

$$\overline{Q} = \frac{\sum_{i=1}^{W_o} Q_i}{W_o}. \tag{12}$$

Again, this formula coincides with its $W\to\infty$ limit and, in addition, it also can be reduced to be smaller than any arbitrarily small quantity ϵ by choosing the parameter δ as appropriately small in size.

The above remarks, establish the stability conditions for the $q-$mean value for a class of disturbances which are closely similar to the ones employed in Ref. [7] , but slightly changed in order to satisfy the normalization conditions for the *escort* probabilities in the $W\to\infty$ limit.

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